stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE ZW 151/81 MAART
(DEPARTMENT OF PURE MATHEMATICS)

A.E. BROUWER & A. NEUMAIER

STRONGLY REGULAR GRAPHS WHERE MU EQUALS TWO AND LAMBDA IS LARGE

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematical subject classification: 05B30

Strongly regular graphs where mu equals two and lambda is large

by

A.E. Brouwer & A. Neumaier*)

ABSTRACT

Certain strongly regular graphs with $\mu = 2$ are killed by observing that the first constituents are partial linear spaces.

KEY WORDS & PHRASES: *strongly regular graphs*

---

*) Insitut für Angewandte Mathematik
Universität Freiburg, D-7800 Freiburg, BRD

0. INTRODUCTION

Let G be a strongly regular graph with parameters v,k,λ,μ where $\mu = 2$. Let x be a fixed vertex, and $H = \Gamma(x)$ the graph induced on the neighbours of x.

Then H has k vertices, is regular of valency λ and two nonadjacent vertices have at most one common neighbour. Consequently each edge of H is contained in a unique maximal clique, and H is the pointgraph of a partial linear space $(H,\mathcal{L})$ which has the property that if $y \in H$, $L \in \mathcal{L}$ and $y \notin L$ then y is on at most one line intersecting L. [i.e., we have a partial quadrangle where lines do not necessarily have constant size.]

Looking at the structure of H one can kill graphs that just escape the CLAW bound: for example, if $\mu = 2$ and $s = -4$ then $f = 24\lambda + \frac{210}{\lambda+6}$ so that $\lambda \in \{0,1,4,8,9,15,24,29,36,64,99,204\}$. The CLAW bound kills the graphs with $\lambda \geq 24$. Here we shall show that also $\lambda \in \{8,9,15\}$ is impossible, thus leaving the parameter sets $(v,k,\lambda) = (56,10,0)$, $(99,14,1)$, $(300,26,4)$. The first one corresponds to the GEWIRTZ graph, the other two are unknown.

1. PARTIAL LINEAR SPACES OF GIRTH 5

In view of the application to strongly regular graphs we shall use k for the number of points and λ for the valency (of the pointgraph) of a partial linear space.

THEOREM. A connected partial linear space with girth at least 5 and more than one line (lines possibly of varying size) in which every point has λ neighbours, contains $k \geq \frac{1}{2}\lambda(\lambda+3)$ points.

PROOF. Let L be a line of size ℓ. Denote by T the set of points at distance at least two from L. Then $|T| = k-\ell(\lambda+2-\ell)$, and $\ell \leq \lambda$ since a line of size $\lambda+1$ would be a component. Let x_i be the number of points in T having exactly i neighbours at distance one from L. We have

(i) $\sum_i x_i = k - \ell(\lambda+2-\ell)$,

(ii) $\sum_i ix_i \geq \ell(\lambda+1-\ell)(\ell-1)$,

(iii) $\sum_i \binom{i}{2}x_i \leq \binom{\ell}{2}(\lambda+1-\ell)^2$.

Hence

$$0 \leq \sum_i (i-(\lambda+2-\ell))^2 x_i$$

$$= 2\sum_i \binom{i}{2}x_i - (2\lambda+3-2\ell)\sum_i ix_i + (\lambda+2-\ell)^2 \sum_i x_i$$

$$\leq \ell(\ell-1)(\lambda+1-\ell)^2 - \ell(\ell-1)(\lambda+1-\ell)(2\lambda+3-2\ell) + (\lambda+2-\ell)^2(k-\ell(\lambda+2-\ell)),$$

whence

$$k(\lambda+2-\ell)^2 \geq \ell(\lambda+2-\ell)((\lambda+2)(\lambda+1-\ell)+1)$$

which can be written as

$$k \geq \tfrac{1}{2}\lambda(\lambda+3) + \frac{(\lambda+2)(\lambda-\ell)(2\ell-3-\lambda)}{2(\lambda+2-\ell)}$$

It follows that if there is a line of size ℓ with $\ell \geq \frac{1}{2}(\lambda+3)$ then $k \geq \frac{1}{2}\lambda(\lambda+3)$, (with strict inequality unless $\ell = \lambda$ or $\ell = \frac{1}{2}(\lambda+3)$).

If ℓ is relatively small then we can imporve on estimate (ii). Let m be the size of the longest line intersecting L. We have

(ii)' $\sum_i ix_i \geq \ell(\lambda+1-\ell)(\lambda+1-m)$.

Hence, evaluating $0 \leq \sum_i (i-\ell)(i-\ell-1)x_i$ we find

$$k \geq (\lambda+1)^2 - \ell\lambda - \frac{2\ell(m-\ell)(\lambda+1-\ell)}{\ell+1}$$

It follows that if the longest line in our partial linear space has length at most $\frac{1}{2}(\lambda+1)$ (putting $m \leq \ell \leq \frac{1}{2}(\lambda+1)$) then $k \geq \frac{1}{2}(\lambda+1)(\lambda+2) = \frac{1}{2}\lambda(\lambda+3)+1$.

In case the longest line has length $\frac{1}{2}(\lambda+2)$, we have to estimate somewhat more carefully. If there is a line L of length $\ell \le \frac{1}{2}\lambda$ such that each line intersecting L has length at most $\frac{1}{2}\lambda$ then $k \ge \frac{1}{2}\lambda^2 + 2\lambda + 1$. This shows that for smaller k there are many lines of size $\frac{1}{2}\lambda + 1$; in fact too many.

Write $|M| = 1+s_M$ for each line M. Considering the lines M distinct from L passing through a point $x \in L$ we see that x is at distance two from $\sum s_M(\lambda - s_M)$ points in T. But $\sum s_M = \lambda + 1 - \ell$, so

$$\sum s_M(\lambda-s_M) = \lambda(\lambda+1-\ell) - (\lambda+1-\ell)^2 + \sum_{M\neq N} s_M s_N$$

$$\ge (\ell-1)(\lambda+1-\ell) + (n_x-1)(2(\lambda+1-\ell)-n_x)$$

where $n_x = \sum_M 1$ is the number of lines intersecting L in the point x.

Let $n_x = 1$ for j points of L, so that $n_x \ge 2$ for the remaining $\ell - j$ points of L. Then

(ii)'' $\sum ix_i \ge \ell(\ell-1)(\lambda+1-\ell) + 2(\ell-j)(\lambda-\ell)$.

In particular, for $\ell = \frac{1}{2}\lambda+1$ we find, evaluating $0 \le \sum(i-\ell)(i-\ell+1)x_i$, that

$$k \ge \tfrac{1}{2}\lambda^2 + \lambda + 1 + \frac{4(\ell-2)(\ell-j)}{\ell}.$$

On the other hand, if for some $j \in \mathbb{N}$ each line of size $\ell = \frac{1}{2}\lambda+1$ intersects at least j+1 others of this size then considering j(j+1) lines of size ℓ intersecting such lines intersecting a given line L we find $|T| = k - \ell^2 \ge \frac{1}{2}j(j+1)(\ell-1) = \frac{1}{4}\lambda j(j+1)$.

This shows that if the linear space contains lines of size $\ell = \frac{1}{2}\lambda+1$ then

(*) $$k \ge \max_{0\le j\le\ell} \min \{\tfrac{1}{2}\lambda^2 + 3\lambda - 3 - j\,\frac{4\ell-8}{\ell},\ \tfrac{1}{4}\lambda j(j+1) + \tfrac{1}{4}\lambda^2 + \lambda + 1\}.$$

Hence, for $\lambda = 2,4,6,8,10$ we find $k \ge 5,15,27,46,67$ respectively, and in general putting $j = \lceil\sqrt{\lambda}\rceil$ we find $k > \frac{1}{2}\lambda(\lambda+3)$ for $\lambda > 6$. This proves the theorem. □

<u>REMARK</u>. Equality holds in the theorem iff $\lambda = 2$ and $k = 5$. (This partial linear space exists - it is a pentagon.) For: if there is a line of size

$\frac{1}{2}\lambda + 1$ then $k > \frac{1}{2}\lambda(\lambda+3)$ unless $\lambda = 2$ or $\lambda = 6$. But if $\lambda = 6$ and $k = 27$ one sees that each line of size 4 intersects exactly three others and that each point in T has three or four neighbours at distance one of L (where $\ell = 4$) - hence lines of size 4 do not intersect in T and $|T| \geq 18$, $k \geq 34$, contradiction.

Hence there are no lines of size $\frac{1}{2}\lambda+1$, but each line of size at most $\frac{1}{2}(\lambda+1)$ intersects a longer line, so there are lines of size $\frac{1}{2}(\lambda+3)$ or λ. In the former case ($\ell = \frac{1}{2}(\lambda+3)$) we may suppose $\lambda > 3$. We see that $j = \ell$, i.e., each line of size $\frac{1}{2}(\lambda+3)$ intersects only lines of size $\frac{1}{2}(\lambda+1)$. Let there to be a lines of size $\frac{1}{2}(\lambda+3)$. Then there are $\frac{1}{2}(\lambda-a)(\lambda+3)$ points not in one of these lines, and $\frac{1}{4}a(\lambda-1)(\lambda+3)$ incidences of such points with $\frac{1}{2}(\lambda+1)$-lines. But each point is in at most two $\frac{1}{2}(\lambda+1)$-lines, so $(\lambda-a)(\lambda+3) \geq \frac{1}{4}a(\lambda-1)(\lambda+3)$ and $a \leq 3$. If $a \geq 1$ then we find a line L with $\ell = \frac{1}{2}(\lambda+1)$ intersecting only one line of size $\frac{1}{2}(\lambda+3)$, i.e., with $j = 1$. From $0 \leq \sum(i-\ell-1)^2 x_i$ one finds $\ell \leq 1$, contradiction.

Hence there are no lines of size $\frac{1}{2}(\lambda+3)$ and all lines have size 2 or λ.

If some point is only in lines of size 2 then it has λ neighbours and $\lambda(\lambda-1)$ points at distance 2 so that $k \geq \lambda^2+1$. On the other hand, if $\lambda > 2$ and each point is in a line of size λ, then let L be a line of size λ. Each of its λ neighbours is in a line of length λ, and these lines cannot intersect, so $|T| \geq \lambda(\lambda-1)$ and $k \geq \lambda(\lambda+1)$. This proves our claim.

<u>REMARK</u>. We do not know the right order of magnitude of the lower bound. The theorem gives something of order $\frac{1}{2}\lambda^2$ - on the other hand, the Moore graphs of diameter two are examples with $k = \lambda^2 + 1$. For small λ we have:

$\lambda = 2$, $k = 5$, the pentagon.
$\lambda = 3$, $k = 10$, the Petersen graph.
$\lambda = 4$, $k = 15$:

There is a unique partial linear space on 15 points with 10 lines of size 3 and girth 5. Its point-graph is distance regular with parameters $i(4,2,1;1,1,4)$. It is obtained from the generalised quadrangle GQ(2,2) by deleting a parallel class of lines. It is the line graph of the Petersen graph.

Now these three examples are regular: all lines have the same size. But for regular partial linear spaces these same methods yield stronger bounds: we have

$$k \geq \lambda(\lambda-\ell+2) + 1$$

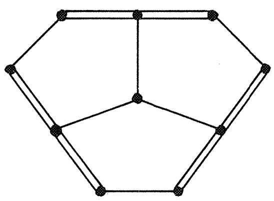
if all lines have size ℓ. If $\lambda \leq \ell(\ell-1)$ this can be strengthened to

$$k \geq \ell^2(\lambda-2\ell+3) + \frac{\ell(\ell-1)^3}{\lambda} .$$

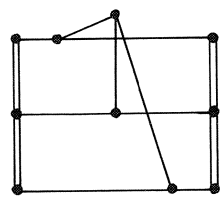
(In case $\lambda = \ell(\ell-1)$, equality would mean that we have a strongly regular graph with $\mu = 1$ and discriminant $(\ell-1)\sqrt{5}$, hence equality occurs only for $\ell = 2$. In general equality means that we have a strongly regular graph with $\mu = 1$ in the first case and distance regular graph of diameter at most three in the second case - the linegraph of a system satisfying one bound with equality, satisfies the other with equality. This yields very strong conditions on the parameters, and only finitely many examples are known.)

An infinite family of regular examples if provided by the incidence graphs of finite projective planes: they have $k = 2(\lambda^2-\lambda+1)$, for $\lambda = q+1$, q a primepower.

Concerning irregular examples there are precisely two others with $\lambda = 3$, $k = 10$ namely



and



For $\lambda = 2$, $k = 5$ and $\lambda = 4$, $k = 15$ there are no irregular examples. There are no examples with $\lambda = 5$, $k = 21$.

2. STRONGLY REGULAR GRAPHS WITH $\mu = 2$.

As already remarked in the introduction, if G is any graph such that two nonadjacent vertices have at most two common neighbours then the collection of neighbours of a fixed vertex x_0 carries a partial linear space of girth at least 5. Now if G is moreover regular and edge-regular then if not $k \geq \frac{1}{2}\lambda(\lambda+3)$ then by the theorem $\Gamma(x_0)$ is a disjoint union of lines of size $\lambda + 1$. It follows that $(\lambda+1)|k$ and that G itself is a partial linear space. Thus:

COROLLARY. *A strongly regular graph with* $\mu = 2$ *and* $k < \frac{1}{2}\lambda(\lambda+2)$ *is a partial quadrangle; in particular it satisfies the divisibility condition* $(\lambda+1)|k$.

This corollary rules out infinitely many feasible sets of parameters of strongly regular graphs e.g. $(v,k,\lambda) = (736,42,8)$, $(875,46,9)$ and $(1961,70,15)$, the three cases mentioned in the introduction. Each of these is the first in an infinite series, e.g. $(v,k,\lambda) = (t^2(6t+11)(3t+2), 6t^2+10t-2, 5t-2)$ is admissible for $t \equiv 1,2,4$ or $6 \pmod 7$ but excluded by the corollary for $t \geq 2$.

Looking at the table we found one parameter set that just escaped the bound, namely

$$(v,k,\lambda) = (1944,67,10).$$

But also this one is easily killed. Returning to the proof of the theorem in the special case $k = 67$, $\lambda = 10$ we see that there are no lines of size 7,8 or 9 and that each line of length 6 intersects at least 4 and hence exactly 4 other lines of size 6. If there is a line of size 6 then there are at least $1 + 4 + 12 = 17$ such lines, together containing at least $(2 + 4.\frac{1}{2}).17 = 68$ points, a contradiction.

Hence there are only lines of size 2 or 10 and $k \geq \lambda^2 + 1 = 101$, again a contradiction.

Clearly our result can also be applied to other distance regular graphs, but we have no examples at present.

strongly regular graphs with 1 <= n <= 2000 and mu=2 but not with the parameters of a net (OA)

					parameters of a net (OA)				
!	n=4	k=2	lb=0	mu=2	complete bipartite graph				
!	n=16	k=5	lb=0	mu=2	r=1	s=-3	f=10	g=5	CLEBSCH
					Hub C12-(2,2)		vL-S(1)	TWO	
		k=10	lb=6	mu=6	r=2	s=-2	f=5	g=10	Kag(4)
					vL-S(2)	TWO			
!	n=56	k=10	lb=0	mu=2	r=2	s=-4	f=35	g=20	GEWIRTZ Hub S
					Quasi-symmetric design				
		k=45	lb=36	mu=36	r=3	s=-3	f=20	g=35	
	n=85	k=14	lb=3	mu=2	r=4	s=-3	f=34	g=50	
		k=70	lb=57	mu=60	r=2	s=-5	f=50	g=34	
	n=99	k=14	lb=1	mu=2	r=3	s=-4	f=54	g=44	spg(3,7,1,2)?
		k=84	lb=71	mu=72	r=3	s=-4	f=44	g=54	
+	n=243	k=22	lb=1	mu=2	r=4	s=-5	f=132	g=110	
					BERLEKAMP-SEIDEL-vanLINT				spg(3,11,1,2)
		k=220	lb=199	mu=200	r=4	s=-5	f=110	g=132	
	n=300	k=26	lb=4	mu=2	r=6	s=-4	f=117	g=182	
		k=273	lb=248	mu=252	r=3	s=-7	f=182	g=117	
	n=352	k=26	lb=0	mu=2	r=4	s=-6	f=208	g=143	
		k=325	lb=300	mu=300	r=5	s=-5	f=143	g=208	
-	n=456	k=35	lb=10	mu=2	r=11	s=-3	f=95	g=360	CLAW
	n=630	k=37	lb=4	mu=2	r=7	s=-5	f=259	g=370	
		k=592	lb=556	mu=560	r=4	s=-8	f=370	g=259	
	n=704	k=37	lb=0	mu=2	r=5	s=-7	f=407	g=296	
		k=666	lb=630	mu=630	r=6	s=-6	f=296	g=407	
-	n=736	k=42	lb=8	mu=2	r=10	s=-4	f=207	g=528	AEB
-	n=875	k=46	lb=9	mu=2	r=11	s=-4	f=230	g=644	AEB
	n=1176	k=50	lb=4	mu=2	r=8	s=-6	f=500	g=675	spg(6,10,1,2)?
		k=1125	lb=1076	mu=1080	r=5	s=-9	f=675	g=500	
	n=1276	k=50	lb=0	mu=2	r=6	s=-8	f=725	g=550	
		k=1225	lb=1176	mu=1176	r=7	s=-7	f=550	g=725	
	n=1625	k=58	lb=3	mu=2	r=8	s=-7	f=754	g=870	
		k=1566	lb=1509	mu=1512	r=6	s=-9	f=870	g=754	
-	n=1944	k=67	lb=10	mu=2	r=13	s=-5	f=536	g=1407	AN & AEB
-	n=1961	k=70	lb=15	mu=2	r=17	s=-4	f=370	g=1590	AEB